Auckland Mathematical Olympiad 2024

Problems with solutions

1. The train consists of six cars. On average, each carriage carries 18 passengers. After one car was uncoupled, the average number of passengers in the remaining cars was reduced to 15. How many passengers were in the uncoupled car?

   Solution. Let x_i be the number of people in the ith car. Then $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 108$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 75$. Hence $x_6 = 33$. $\square$

2. In how many ways can 8 people be divided into pairs?

   Solution. Let people be $\{1, 2, \ldots, 8\}$. Person 1 can be paired with 7 people. For each choice we have 6 people left. Selecting one of them, we see that she can be paired with 5 people. Then 4 people left that can be paired in 3 ways. In total we have $7 \times 5 \times 3 = 105$ ways. $\square$

3. Prove that for arbitrary real numbers a and b the following inequality is true
$$a^2 + ab + b^2 \geq 3(a + b - 1).$$

   Solution. Let us write it as
$$a^2 + (b - 3)a + (b^2 - 3b + 3) \geq 0.$$

   Consider the left-hand-side as a quadratic in a we see that its discriminant d is non-positive. Indeed, $d = (b-3)^2 - 4(b^2 - 3b + 3) = -3b^2 + 6b - 3 = -3(b-1)^2$. Hence this quadratic takes only non-negative values. Many other solutions exist. $\square$

4. The altitude AH and the bisector CL of triangle ABC intersect at point O. Find the angle BAC, if it is known that the difference between angle COH and half of angle ABC is $46°$.

   Answer. $88°$. Let us denote halves of Angles A, B, C by x, y, z, respectively. Then $\angle COH = 90° - z$ and $46° = 90° - z - y$. Since $x + y + z = 90°$, then $x = 44$ and $\angle BAC = 88$.

5. Prove that the number $2^9 + 2^{99}$ is divisible by 100.

   Solution. We have $a + b \mid a^9 + b^9$. Thus $(2 + 2^9)k = 2^9 + 2^{99}$, where k is even. But $2 + 2^9 = 2050$ is divisible by 50. Since k is even, $2^9 + 2^{99}$ is divisible by 100. □

6. There are 50 coins in a row; each coin has a value. Two people are playing a game alternating moves. In one move a player can take either the leftmost or the rightmost coin. Who can always accumulate coins whose total value is at least the value of the coins of the opponent?

   Solution. Let's assume that Player A goes first. First, she colours the coins red and blue so that no two neighbouring coins have the same colour. Suppose red coins in total have a greater value.

   We prove that Player A can get all of the red coins. Her strategy is to simply pick the coin at whichever end of the sequence that is red (there will be one). After doing so, the board will have two blue coins at both ends. No matter what Player B picks, after his turn Player B will reveal a red coin at some end. Player A will then pick that red coin. This goes on until all coins are gone. □

7. There are 20 points marked on a circle. Two players take turns drawing chords with ends at marked points that do not intersect the already drawn chords. The one who cannot make the next move loses. Who can secure their win?

   Solution. The firs player has a winning strategy. without loss of generality we may assume the points arranged in the vertices of a regular 20-gon. The first player draws the chord connecting two opposite points and then mirrors symmetrically moves of the second player. □

8. There are 25 points on the plane, and among any three of them there are two at a distance less than 1. Prove that there is a circle of radius 1 containing at least 13 of these points.

   Solution. Pick arbitrarily one point A out of the given 25 and consider a circle $B(A, 1)$ centred at A and radius 1. If all the remaining points lie inside $B(A, 1)$, there is nothing to prove - B(A,1) is the circle we've been looking for.

Else, there is a point B that lies outside $B(A,1)$. Let $B(B,1)$ be a unit circle centred at B. By definition, the distance between A and B exceeds 1. For any point C, by the given condition, either the distance to A or the distance to B is less than 1. In other words, each of the remaining points belongs to either $B(A,1)$ or $B(B,1)$. By the Pigeonhole Principle, at least 13 lie in the same circle. □

9. 100 students came to a party. The students who did not have friends among other students left the party first. Then those with one friend among remaining students left. Then those with $2, 3, \ldots, 99$ friends among remaining students left. What is the maximal number of students that can still remain at the party after that. (If A is a friend of B, then B is a friend of A).

   Solution. **Answer:** 98. Imagine the situation when every two students are friends apart from students A and B who are not. When it is time for students with 98 friends to go, both A and B will leave and remaining students will have 97 friends each so nobody will leave after that.

   This number 98 is maximal. Suppose 99 students stay at the end and only one, say C, left. Suppose C had k friends. Then $k < 99$. If nobody left after that, then all remaining students would have k friends after C left. But this means that 99 students had C as a friend and $k \geq 99$, a contradiction. □

10. Prove that circles constructed on the sides of a convex quadrilateral as diameters completely cover this quadrilateral.

    Solution. Let $ABCD$ be a convex quadrilateral and X is a pint inside. Consider four angles $\angle AXB$, $\angle BXC$, $\angle CXD$, $\angle DXA$. Their sum is 360°. Hence one of them is at least 90°, say $\angle AXB \geq 90^\circ$. Then X belongs to a circle constructed on AB as diameter. □

11. It is known that for quadratic polynomials $P(x) = x^2 + ax + b$ and $Q(x) = x^2 + cx + d$ the equation

    $$P(Q(x)) = Q(P(x))$$

    does not have real roots. Prove that $b \neq d$.

12. The representation of real number a as a decimal infinite fraction contain all 10 digits. For a positive integer n let v_n be the number of all segments of length n that occur. Prove that, if $v_n \leq n+8$ for some positive integer n, then the number a is rational.

    Solution. We have $v_1 = 10$. If only $v_{n+1} = v_n$ for some n, then every segment of length n has a unique continuation so the sequence is periodic from some point and a is rational. Alternatively,

$$v_n \geq v_{n-1} + 1 \geq \cdots \geq v_2 + n - 2 \geq v_1 + n - 1 = n + 9,$$

    a contradiction. □